

# Quantum limits in interferometric measurements

M.T. Jaekel <sup>a</sup> and S. Reynaud <sup>b</sup>

(a) *Laboratoire de Physique Théorique de l'Ecole Normale Supérieure* \* 24 rue Lhomond F-75231 Paris Cedex 05

(b) *Laboratoire de Spectroscopie Hertzienne* <sup>†</sup> Université Pierre et Marie Curie 4 place Jussieu F-75252 Paris Cedex 05  
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Quantum noise limits the sensitivity of interferometric measurements. It is generally admitted that it leads to an ultimate sensitivity, the “standard quantum limit”. Using a semi-classical analysis of quantum noise, we show that a judicious use of squeezed states allows one in principle to push the sensitivity beyond this limit. This general method could be applied to large scale interferometers designed for gravitational wave detection.

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Quantum noise ultimately limits the sensitivity in interferometric detection of gravitational waves [1–3]. A gravitational wave is detected as a phase difference between the optical lengths of the two arms. It seems accepted that there exists a “standard quantum limit” (SQL), equivalent to an ultimate detectable length variation

$$(\Delta z)_{SQL} = \sqrt{\frac{\hbar \tau}{M}} \quad (1)$$

where  $M$  is the mass of the mirrors and  $\tau$  the measurement time [4]. The SQL can be derived by considering that the positions  $z(t)$  and  $z(t + \tau)$ , which are non commuting observables, are measured [5]. This interpretation of SQL has given rise to a long controversy [6].

Alternatively, the SQL can be understood by considering the quantum noise as a sum of two contributions. Photon counting noise corresponds to fluctuations of the number of photons detected in the two output ports while radiation pressure noise stems from the random motion of the mirrors which is sensitive to the fluctuations of the numbers of photons in each arm. The sum of these two contributions leads to an optimal sensitivity given by the expression (1). This limit is reached for very large laser power which is not presently achievable.

Caves [7] pointed out that these two contributions reflect the fluctuations of two quadrature components of the vacuum field entering the unused input port of the interferometer. He further suggested [8] a reduction in the photon counting noise by entering squeezed light [9]

in this port. This possibility, which has been experimentally demonstrated [10], allows one to attain the optimal sensitivity for more reasonable laser powers. It does not overcome the SQL because the reduction in photon noise is compensated by the increase in radiation pressure fluctuations.

Unruh [11] has shown that a judicious extension of Caves’ proposal leads to a sensitivity beyond the SQL. The key point is that photon counting noise and radiation pressure noise are not independent sources of fluctuations as implicitly assumed in the derivation of the SQL. Both contributions are linearly superimposed in the fluctuations of the monitored signal [12]. It is therefore possible to reduce the total noise by squeezing the appropriate quadrature component of the field entering the unused input port.

We present in this letter a simple method for treating quantum noise in interferometric measurements. It is based on a semi-classical linear input output theory already used for computing the field fluctuations generated by optical parametric oscillators [13]. Using this general method, we show that the sensitivity can be pushed beyond the SQL. The method can be used for incorporating a detailed analysis of quantum noise in discussions about large scale interferometers designed for gravitational wave detection. As a first step in this direction, we discuss the quantum limits when taking into account some constraints.

## MEASUREMENT OF THE POSITION OF A MIRROR

We first analyze the simple case where the position of a single mirror is measured. The mirror is illuminated by an incident laser beam. The phase of the reflected field at some arbitrary point contains the information about the mirror position.

The incident electric field  $\mathbf{E}(t)$  will be written as the sum of a prescribed monochromatic field (frequency  $\omega_0$  and wavevector  $k_0 = \frac{\omega_0}{c}$ ) and the quantum fluctuations of a monodimensional scalar field (propagation along the  $z$  direction with one polarization only, but all possible frequencies)

$$\mathbf{E}(t) = \sqrt{\frac{\hbar \omega_0}{2 \varepsilon_0 c}} (\mathbf{u}(t) e^{ik_0 z - i \omega_0 t} + \mathbf{u}^\dagger(t) e^{-ik_0 z + i \omega_0 t}) \quad (2a)$$

$$\mathbf{u}(t) = \frac{1}{2} (< \mathbf{p} > + \delta \mathbf{p}(t) + i \delta \mathbf{q}(t)) \quad (2b)$$

\*Unité propre du Centre National de la Recherche Scientifique associée à l'Ecole Normale Supérieure et à l'Université de Paris-Sud.

<sup>†</sup>Unité de l'Ecole Normale Supérieure et de l'Université Pierre et Marie Curie, associée au Centre National de la Recherche Scientifique.

$$\delta \mathbf{x}(t) = \mathbf{x} - \langle \mathbf{x} \rangle \quad (2c)$$

The operators  $\mathbf{p}$  and  $\mathbf{q}$  represent respectively the amplitude and phase quadrature components of the field.

We will use a semi-classical description of quantum fluctuations [13]. The classical random variables  $p$  and  $q$  are defined so that they fit the symmetrically ordered momenta of the quantum fluctuations  $\mathbf{p}$  and  $\mathbf{q}$

$$\langle x(t)y(t') \rangle = \frac{1}{2} \langle \mathbf{x}(t)\mathbf{y}(t') + \mathbf{y}(t')\mathbf{x}(t) \rangle \quad (3)$$

They are characterized by the spectra of stationary random variables

$$S_{xy}(\Omega) = \int_{-\infty}^{+\infty} dt \langle \delta x(t_0 + t) \delta y(t_0) \rangle e^{i\Omega t} \quad (4)$$

The spectra obey a generalized Heisenberg inequality [14] which can be written (in the frequency domain  $\Omega \ll \omega_0$ ) as

$$S_{pp}(\Omega)S_{qq}(\Omega) - S_{pq}(\Omega)^2 \succeq 1 \quad (5)$$

Using these notations, we can define a normalized intensity  $I$ , measured as a number of photons per unit time, and a phase  $\varphi$

$$\begin{aligned} u(t) &= \frac{1}{2} (\langle p \rangle + \delta p(t) + i\delta q(t)) \\ &= \sqrt{I + \delta I(t)} \exp(i\delta\varphi(t)) \end{aligned} \quad (6)$$

In a linear treatment of the field fluctuations, one obtains

$$I = \frac{1}{4} \langle p \rangle^2, \quad \delta I(t) = \frac{1}{2} \langle p \rangle \delta p(t) = \sqrt{I} \delta p(t) \quad (7)$$

$$\delta\varphi(t) = \frac{\delta q(t)}{\langle p \rangle} = \frac{\delta q(t)}{2\sqrt{I}} \quad (8)$$

We can now discuss the effect of quantum noise for this position measurement. The phase of the reflected beam can be written

$$\varphi = 2k_0 z + \delta\varphi(t) \quad (9)$$

where  $\delta\varphi(t)$  represents the incident phase fluctuations (8) and where the mirror position  $z$  depends on the intensity fluctuations (7), due to radiation pressure force  $2\hbar k_0 |u(t)|^2$ . In order to evaluate this term, we have to describe the response of the mirror to the force. In a linear analysis, this response is described in the frequency domain by a susceptibility function  $\chi$ .

Thus,  $\frac{\varphi(\Omega)}{2k_0}$  provides an estimator  $\tilde{z}(\Omega)$  for each frequency component of the position

$$\tilde{z}(\Omega) = \bar{z}(\Omega) + \delta\bar{z}(\Omega) \quad (10)$$

where  $\bar{z}(\Omega)$  and  $\delta\bar{z}(\Omega)$  correspond to the signal and noise. The noise is a sum of three error terms associated respectively with the incident phase fluctuations  $\delta z_{pc}$  superimposed to the signal, the mirror displacement  $\delta z_{rp}$  due to

the radiation pressure and the mirror displacement  $\delta z_{ef}$  due to extra fluctuations  $\delta\bar{f}$

$$\delta\bar{z}(\Omega) = \delta\bar{z}_{pc}(\Omega) + \delta\bar{z}_{rp}(\Omega) + \delta\bar{z}_{ef}(\Omega) \quad (11a)$$

$$\delta\bar{z}_{pc}(\Omega) = \frac{\delta\bar{q}(\Omega)}{4k_0\sqrt{I}} \quad (11b)$$

$$\delta\bar{z}_{rp}(\Omega) = \chi(\Omega) 2\hbar k_0 \sqrt{I} \delta\bar{p}(\Omega) \quad (11c)$$

$$\delta\bar{z}_{ef}(\Omega) = \chi(\Omega) \delta\bar{f}(\Omega) \quad (11d)$$

## INTERFEROMETRIC MEASUREMENT

Up to now, we have not discussed a practical realisation of the phase measurement. In fact, this is the role of the interferometer to transform the phase signal into an intensity signal.

The interferometer can be schematized as consisting of two input ports  $A$  and  $B$ , two output ports  $C$  and  $D$  and two internal paths 1 and 2. We will consider here a simple configuration. A mean field is entered only into the port  $A$ . The beam splitters have equal transmission and reflection probabilities and the difference  $J$  between the two output intensities is measured around a working point where it is zero. One obtains in this case

$$\delta I = \delta I_1 - \delta I_2 = \sqrt{I_A} \delta p_B(t) \quad (12a)$$

$$J = I_C - I_D = I_A 2k_0 z + \sqrt{I_A} \delta q_B(t) \quad (12b)$$

where  $z$  is the difference between the optical lengths of the two arms (treated as a small quantity) and where  $\delta p_B$  and  $\delta q_B$  represent the amplitude and phase fluctuations of the field entering the port  $B$  ( $I_A$  is the mean intensity entering the port  $A$ ).

If the two mirrors have the same susceptibility  $\chi$ , the differential displacement  $z$  can be written

$$\bar{z}(\Omega) = \chi(\Omega) (-M\Omega^2 \bar{s}(\Omega) + 2\hbar k_0 \delta\bar{I}(\Omega) + \delta\bar{f}(\Omega)) \quad (13)$$

where the force is the sum of three terms associated respectively with the gravitational wave, the radiation pressure and the extra fluctuations. The gravitational signal is measured as the variation  $s$  of a distance between two free falling mirrors.

A signal estimator  $\tilde{s}(\Omega)$  can be defined at each frequency

$$\tilde{s}(\Omega) = \frac{\bar{J}(\Omega)}{2k_0 I_A} = -M\Omega^2 \chi(\Omega) \bar{s}(\Omega) + \delta\bar{s}(\Omega) \quad (14)$$

where the error is, as before, the sum of three terms

$$\delta\bar{s}(\Omega) = \delta\bar{s}_{pc}(\Omega) + \delta\bar{s}_{rp}(\Omega) + \delta\bar{s}_{ef}(\Omega) \quad (15a)$$

$$\delta\bar{s}_{pc}(\Omega) = \frac{\delta\bar{q}_B(\Omega)}{2k_0\sqrt{I_A}} \quad (15b)$$

$$\delta\bar{s}_{rp}(\Omega) = \chi(\Omega) 2\hbar k_0 \sqrt{I_A} \delta\bar{p}_B(\Omega) \quad (15c)$$

$$\delta\bar{s}_{ef}(\Omega) = \chi(\Omega) \delta\bar{f}(\Omega) \quad (15d)$$

Quantum fluctuations and extra fluctuations are independent, but  $p$  and  $q$  fluctuations can be correlated. One obtains from (15) the noise spectrum

$$S_{ss}(\Omega) = \frac{S_{qq}(\Omega)}{4k_0^2 I_A} + 2\hbar\chi_R(\Omega)S_{pq}(\Omega) + (\chi_R^2(\Omega) + \chi_I^2(\Omega)) (4\hbar^2 k_0^2 I_A S_{pp}(\Omega) + S_{ff}(\Omega)) \quad (16)$$

where  $\chi_R(\Omega)$  and  $\chi_I(\Omega)$  are the real and imaginary parts of  $\chi(\Omega)$  (the spectra  $S_{qq}$ ,  $S_{pq}$  and  $S_{pp}$  refer to the input port  $B$ ).

We will consider that the signal is measured through a filter characterized by a function  $G(\Omega)$  (maximum value 1 at the signal frequency  $\Omega_S$ ). The filtered noise is given by the integral

$$\Delta s^2 = 2B\overline{S_{ss}} \quad (17)$$

where  $B$  is the detection bandwidth

$$2B = \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi} G(\Omega) \quad (18)$$

and where  $\overline{F}$  is the mean value of a function  $F$  over the normalized frequency distribution  $\frac{G}{4\pi B}$ .

When the mirrors are held in their equilibrium positions by damped harmonic systems, the susceptibility is

$$\chi(\Omega) = \frac{1}{M(\Omega_M^2 - \Omega^2 - i\Gamma\Omega)} \quad (19)$$

and the signal is accurately reproduced when the eigenfrequency  $\Omega_M$  and the damping constant  $\Gamma$  are smaller than the signal frequency  $\Omega_S$ . More general expressions of  $\chi(\Omega)$ , which are studied for the large scale interferometers, must also obey [2]

$$-M\Omega^2\chi(\Omega) \sim 1 \quad \text{for} \quad \Omega_S - 2B < \Omega < \Omega_S + 2B \quad (20)$$

## QUANTUM LIMITS

In the following, we disregard the extra fluctuations and consider only the quantum noise

$$\Delta s^2 = \frac{2B}{4k_0^2 I_A} \overline{S_{qq}} + 4B\hbar \overline{\chi_R S_{pq}} + 8B\hbar^2 k_0^2 I_A \overline{(\chi_R^2 + \chi_I^2) S_{pp}} \quad (21)$$

We first consider the simplest case where the fluctuations entering the port  $B$  are vacuum fluctuations ( $S_{pp}(\Omega) = S_{qq}(\Omega) = 1$ ;  $S_{pq}(\Omega) = 0$ )

$$\Delta s^2 = \frac{2B}{4k_0^2 I_A} + 8B\hbar^2 k_0^2 I_A \overline{\chi_R^2 + \chi_I^2} \quad (22)$$

In this case, the quantum noise is effectively the sum of two independent contributions associated with phase

and amplitude fluctuations. Its minimum when the laser intensity  $I_A$  is varied is the standard quantum limit

$$\Delta s_{SQL}^2 = 4B\hbar \sqrt{\chi_R^2 + \chi_I^2} \quad (23)$$

Using condition (20), we obtain the usual expression (1) with a time parameter  $\tau \approx \frac{4B}{\Omega_S^2}$ .

Caves' proposal [8] corresponds to squeezed phase fluctuations ( $S_{pp}(\Omega) = K$ ;  $S_{qq}(\Omega) = \frac{1}{K}$ ;  $S_{pq}(\Omega) = 0$ ). It also leads to the standard quantum limit (23), but for a smaller laser intensity.

We finally consider Unruh's proposal [11] where correlated squeezed fluctuations enter the input port  $B$ . The noise  $\Delta s^2$  can be decreased below the SQL by varying  $S_{qq}(\Omega)$ ,  $S_{pq}(\Omega)$  and  $S_{pp}(\Omega)$  and respecting the Heisenberg inequality (5). A lower bound for the sensitivity is obtained by assuming that the squeezing can be optimised at each frequency. One finds in this way

$$\Delta s^2 = 4B\hbar |\overline{\chi_I}| \quad (24)$$

This lower bound is far below the standard quantum limit (23) since the reactive part  $\chi_R$  of the susceptibility is much larger than the dissipative part  $\chi_I$  when the condition (20) is satisfied. For example, the damped harmonic system leads to the expression (1) with a time parameter  $\tau \approx \frac{4B\Gamma}{\Omega_S^3}$ . It has to be noted that the recoil effect associated with the reflection of photons [11] gives rise to a damping  $\Gamma_{min} = \frac{\hbar k_0 I_A}{M}$ . In practice, the damping constant is larger than this minimum value but the lower bound (24) is still below the SQL (23) as long as  $\Gamma < \Omega_S$ .

If broadband correlated squeezing is used, one has another minimum noise

$$\Delta s^2 = 4B\hbar \sqrt{\chi_R^2 + \chi_I^2 - \overline{\chi_R}^2} \quad (25)$$

This noise is intermediate between the SQL (23) and the lower bound (24). It can be shown that it is close to the SQL (23) when condition (20) is satisfied. This shows that it is important to control the squeezing parameters at each frequency in order to approach the lower bound (24).

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